Lemma 5.1: Let (X,Δ) be a lc pair & let (X,Φ) be dlt pair, where X is $\mathbb{Q}$-factorial of dim n. Let $\Delta(t) = (1-t)\Delta + t\Phi$. Suppose that $\pi: X \to U$ is projective, equidimensional, U is smooth & affine. Let $f: X \dashrightarrow Y$ be a step of the $(K_X + \Delta(t))$-MMP$/U$ & let $\Gamma = f_*\Delta$. Suppose $0 \in U$ is a closed pt s.t X_0 is reduced, $X_0 \not\subset \operatorname{Supp}\Delta$, $K_{X_0} + \Delta_0$ is nef & (X_0, Δ_0) is lc. Let r be a +ve integer s.t $r(K_{X_0} + \Delta_0)$ is Cartier. If $0 < t \leq \frac{1}{1+2nr}$, then f is $(K_X + \Delta)$-trivial in a nbd of X_0. In particular, (Y_0, Γ_0) is lc, $K_{Y_0} + \Gamma_0$ is nef, $r(K_{Y_0} + \Gamma_0)$ is Cartier & (Y,Γ) is lc in a nbd of Y_0.

Pf: Let R be the extremal ray corresponding to f. Then $(K_{X_0} + \Delta_0)\cdot R = 0$
Let $(K_X + \Delta)\cdot R = 0$. Suppose f is a flip & consider the flip diagram (Similar arguments work if f is a div cont)

(Put $_0$ everywhere)

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] \arrow[dr, "\varphi = \mathrm{cont}_R"'] & & Y = X^+ \arrow[dl, "\varphi^+"] \\ & Z & \end{tikzcd}$$

& let W be a resolution. Write

$p^*(K_X+\Delta) = q^*(K_Y+\Gamma) + E$ where E is q-exc

Since $(K_X+\Delta)\cdot R = 0$. $K_X+\Delta = \varphi^* L_Z$.

Then $p^*(K_X+\Delta) = p^*\varphi^* L_Z = q^*(\varphi^+)^* L_Z$

& hence $q^*(\varphi^+)^* L_Z = q^*(K_Y+\Gamma) + E$. Thus $E = 0$. & $p^*(K_X+\Delta) = q^*(K_Y+\Gamma)$ $\therefore$ $K_Y+\Gamma$ is also nef.

Otherwise, as $K_{X_0}+\Delta_0$ is nef, $(K_X+\Delta)\cdot R \equiv (K_{X_0}+\Delta_0)\cdot R > 0$

This is b/c R is spanned by a rational curve C contained in X_0. Note $(K_X+\Delta(t))\cdot R < 0 \Rightarrow (K_X+\Phi)\cdot R < 0$ & C satisfies $-(K_X+\Phi)\cdot C \leq 2n$. As $r(K_{X_0}+\Delta_0)$ is Cartier, $(K_X+\Delta)\cdot C = (K_{X_0}+\Delta_0)\cdot C \geq \frac{1}{r}$.

Thus
$$\begin{aligned} 0 > (K_X+\Delta(t))\cdot C &= (1-t)(K_X+\Delta)\cdot C + t(K_X+\Phi)\cdot C \\ &\geq (1-t)/r - 2nt \\ &= \frac{1}{r} - \frac{t(1+2nr)}{r} \geq 0 \text{ (by the hyp on } t\text{)} \end{aligned}$$

This is a contradiction. $\therefore$ $(K_X+\Delta)\cdot R = 0$ always.

Lemma 5.2 : Let $(X, \Delta = S + B)$ be a dlt pair, where $S \subseteq \lfloor \Delta \rfloor$ ("$=$" case will be used later), where X is $\mathbb{Q}$-factorial. Let $\pi : X \to U$ as above. $0 \in U$ closed s.t. X_0 is integral. Let $n = \dim X$ & let $r \in \mathbb{N}$ s.t. $r(K_{X_0} + \Delta_0)$ is Cartier. Suppose $X_0 \not\subset \operatorname{Supp} \Delta$. Fix $\varepsilon < \frac{1}{2nr+1}$. If (X_0, Δ_0) is lc & $K_{X_0} + \Delta_0$ is nef but $K_X + (1-\varepsilon)S + B$ is not pseudoeff, then we may run the $(K_X + (1-\varepsilon)S + B)$-MMP/$U$ $f : X \dashrightarrow Y$, the steps of which are all $(K_X + \Delta)$-trivial in a nbd of X_0 until we get a MFS $\psi : Y \to Z/U$ s.t. the strict transform of S dominates Z & $K_Y + \Gamma \sim_{\mathbb{Q}} \psi^* L$ for some divisor L on Z.

Pf: Run the $(K_X + \underset{= K_X + (1-\varepsilon)\Delta + \varepsilon B}{(1-\varepsilon) S + B})$-MMP $f : X \dashrightarrow Y$ with ample scaling $/U$. Then 5.1 $\Rightarrow$ each step of this MMP is $(K_X + \Delta)$-trivial in a nbd of X_0. As $(K_X + (1-\varepsilon)S + B)$ isn't psef, this MMP ends with a MFS $\psi : Y \to Z$. $(K_X + \Delta)$-triviality of this MMP $\Rightarrow$ $\underset{= K_Y + S_Y + B_Y}{K_Y + \Gamma} \sim_{\psi} 0$, $-(K_Y + (1-\varepsilon)S_Y + B_Y) \sim_{\psi} A$ ample$/Z$.

$\Rightarrow \varepsilon S_Y \sim_{\psi} A$ $\therefore$ S_Y dominates Z. $\square$

Lemma 5.3 Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, X proj. & Δ a $\mathbb{Q}$-divisor. If Φ is a $\mathbb{Q}$-divisor s.t.

$$0 \leq \Delta - \Phi \leq N_\sigma(K_X + \Delta),$$

then (X, Φ) has a good MM iff (X, Δ) does.

Pf: Suppose $f: X \dashrightarrow Y$ (putting $t = 0$ there) is a good MM of (X, Δ). Let $\Gamma = f_* \Delta$. Then by (2) of Lemma 2.7.2, f contracts every cpt of $N_\sigma(K_X + \Delta)$, $\therefore f_* \Delta = f_* \Phi$ $\therefore f_*(K_X + \Delta) = f_*(K_X + \Phi) = K_Y + \Gamma$. Let $W \xrightarrow{p} X$ & $W \xrightarrow{q} Y$ resolve f. If we write $p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E$, then $E \geq 0$ is q-exc & (by nefness of $K_Y + \Gamma$), $p_* E = N_\sigma(K_X + \Delta)$. It follows that if we write $p^*(K_X + \Phi) = q^*(K_Y + \Gamma) + F$, then $F = E - p^*(\Delta - \Phi) \geq E - p^*(N_\sigma(K_X + \Delta)) = E - p^* p_* E$: p-exc. $\therefore p_* F \geq 0$. Then $F \geq 0$ by negativity lemma. So f is a wlc model of (X, Φ). Since f is a good MM of (X, Δ), f is a semiample model of (X, Φ) & so (X, Φ) has a good MM by 2.9.1.

<u>Converse</u>: There seems to be an issue with the proof of converse direction in HMX's lemma 5.3.

Let $f: X \dashrightarrow Y$ be a good MM of $K_X + \Phi$. Let $\Delta(t) = \Phi + t(\Delta - \Phi) = t\Delta + (1-t)\Phi$. Then f is also a run of the $(K_X + \Delta(t))$-MMP for all $0 < t$ suff small. We continue to run the $(K_X + \Delta(t))$-MMP with ample scaling to get to $g: X \overset{f}{\dashrightarrow} Y \overset{h}{\dashrightarrow} W$ s.t g contracts all components of $N_\sigma(K_X + \Delta(t))$ (Don't need to make t any smaller here). Every step of the MMP h is $(K_Y + \Phi_Y)$-trivial, hence h is an MMP/Z where $Y \to Z$ is the ample model of $K_Y + \Phi_Y$.

Now [HMX] claim that components of $N_\sigma(K_X + \Delta(t))$ are the same as those of $N_\sigma(K_X + \Delta)$ (Actually, what's true is that any component of $N_\sigma(K_X + \Delta)$ will be a component of $N_\sigma(K_X + \Delta(t))$ for t sufficiently close to 1. However, for us, t is close to 0. How to fix this?) Assuming this, $N_\sigma(K_X + \Delta)$ is contracted $\therefore g_*(K_X + \Delta) = g_*(K_X + \Phi) = h_*(K_Y + \Phi_Y)$: semiample $\because h$ is $K_Y + \Phi_Y$-trivial. Since g only contracts components of $N_\sigma(K_X + \Delta)$ (again, this is not quite true, components of $N_\sigma(K_X + \Delta(t))$ could be more than those of $N_\sigma(K_X + \Delta)$) 2.7.3 $\Rightarrow g$ is a good MM of (X, Δ) (Probably, need to use 2.9.1 instead).

6. Abundance in families

Lemma 6.1: Suppose that (X,Δ) is a log pair, where coeff$(\Delta) \in (0,1] \cap \mathbb{Q}$. Let $\pi: X \to U$ projective morphism to a smooth variety U. Suppose that (X,Δ) is log smooth over U. If $\exists\, 0 \in U$ closed pt s.t. (X_0, Δ_0) has a good minimal model, then the generic fiber (X_η, Δ_η) has a good minimal model.

Pf: (By 2.9.3, ETS: $(X_{\bar\eta}, \Delta_{\bar\eta})$ has a good MM) Replacing U by a finite cover (Stein factorisation), wma π is a contraction & the strata of Δ have irred fibres over U.

Let $f_0 : Y_0 \to X_0$ be the birational morphism given by 2.8.3 (ie f_0 blows up strata of Δ_0; if $K_{Y_0} + \Gamma_0 = f_0^*(K_{X_0} + \Delta_0) + E_0$ & if $\Gamma_0' \doteq \Gamma_0 - \Gamma_0 \wedge N_\sigma(K_{Y_0} + \Gamma_0)$, then $B_-(K_{Y_0} + \Gamma_0')$ contains no strata of Γ_0'). f_0 extends to a birational morphism $f: Y \to X$ which is a composition of smooth blowups of strata of Δ.

Write $K_Y + \Gamma = f^*(K_X + \Delta) + E$ where $\Gamma, E \geq 0$ have no common components, $f_*\Gamma = \Delta$, $f_* E = 0$. Then (Y,Γ) is

log smooth & the fibers of the cpts of Γ are irreducible.
[HX-2.10]: (Y_0, Γ_0) has a good MM as (X_0, Δ_0) does &
that ETS: (Y_η, Γ_η) has a good MM.

Replacing (X, Δ) by (Y, Γ), w.m.a that if
$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(K_{X_0} + \Delta_0)$, then $B_-(K_{X_0} + \Theta_0)$ contains no strata of Θ_0. Now, $\exists!$ divisor $0 \le \Theta \le \Delta$ s.t $\Theta|_{X_0} = \Theta_0$. Then 2.3.3
$\Rightarrow$ $\Theta = \Delta - \Delta \wedge N_\sigma(K_X + \Delta)$, so we have $\Delta - \Theta \le N_\sigma(K_X + \Delta)$.
Now by 5.3, ETS: (X_η, Θ_η) has a good minimal model.
Replacing (X, Δ) by (X, Θ), w.m.a $B_-(K_{X_0} + \Delta_0)$ contains no strata of Δ_0. Then (3.2) $\Rightarrow$ we can run the $(K_X + \Delta)$-MM-P $f: X \dashrightarrow Y$ over U to obtain a minimal model of the generic fiber. NB: It's good. Let $\Gamma = f_* \Delta$.

Pick a cpt D of $\lfloor \Delta \rfloor$. Let $\phi: D \dashrightarrow E$ be $f|_D$.
Then by (3.2), ϕ_0 is a semiample model of $(D_0, (\Delta_0 - D_0)|_{D_0})$
& then (2.9.1) $\Rightarrow$ $(D_0, (\Delta_0 - D_0)|_{D_0})$ has a good M.M. By induction on the dimension, $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$ has a good MM. (But then $\phi_\eta: D_\eta \dashrightarrow E_\eta$ is a semiample model of

$(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$.

Let $S = \lfloor \Delta \rfloor$ & $B = \{\Delta\} = \Delta - S$. Let $T = f_* S$ & $C = f_* B$. Suppose that $K_{Y_0} + (1-\varepsilon) T_0 + C_0$ isn't psef $\forall\, \varepsilon > 0$. Then $K_{X_0} + (1-\varepsilon) S_0 + B_0$ isn't psef either. This implies that $K_X + (1-\varepsilon) S + B$ isn't psef $\forall\, \varepsilon > 0$. But then $K_Y + (1-\varepsilon) T + C$ isn't psef either. Thus (5.2) says (which is $K_Y + \Gamma$ - trivial in a nbd of Y_0) the following: We may run the $(K_Y + (1-\varepsilon) T + C)$-MMP/A until we get to a MFS $g: Y \dashrightarrow W$, $\psi: W \to V$ over U s.t. $g_*(K_Y + \Gamma) \sim \psi^* L$ for some div L on V.

We know that $K_W + T_W + C_W$ is nef (possibly shrinking U) & $-(K_W + (1-\varepsilon) T_W + C_W)$ is ample $/V$ (Here $T_W = g_* T$, $C_W = g_* C$) $\therefore$ T_W is ample $/V$ (after shrinking U) hence dominates V. We can thus pick a cpt D of S whose image F in W dominates V. $\varphi_\eta : D_\eta \dashrightarrow E_\eta$ is a semiample model of $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$ as observed above $\therefore (\psi^* L)|_{F_\eta}$ is semiample & hence L_η is semiample. Thus the composition $X_\eta \dashrightarrow W_\eta$ is a semiample model of (X_η, Δ_η) $\therefore$ (X_η, Δ_η) has a good minimal model by 2.9.1.

The other possibility is $K_{Y_0} + (1-\varepsilon)T_0 + C_0$ is psef for some $\varepsilon > 0$. If $Y_0 \longrightarrow Z_0$ is the lc model of (Y_0, Γ_0), then T_0 doesn't dominate Z_0, so if $\varepsilon \ll 1$, then $\kappa(K_{X_0} + (1-\varepsilon)S_0 + B_0) = \kappa(K_{X_0} + \Delta_0)$ (This isn't clear) We have

$$\kappa(K_{X_n} + \Delta_n) \geq \kappa(K_{X_n} + (1-\varepsilon)S_n + B_n) \underset{\text{(inv. of p.g.)}}{\overset{4.2}{=}} \kappa(K_{X_0} + (1-\varepsilon)S_0 + B_0)$$

$$= \kappa(K_{X_0} + \Delta_0) = \kappa_\sigma(K_{X_0} + \Delta_0) = \nu(K_{Y_0} + \Gamma_0) \underset{\text{inv. of int \#s.}}{\overset{\text{def}}{=}} \nu(K_{Y_n} + \Gamma_n) \; (*)$$

We've seen that if E is a cpt of T, then $(K_Y + \Gamma)|_{E_n}$ is semiample $\therefore$ $(K_Y + \Gamma)|_{T_n}$ is semiample.

Let $H = K_{Y_n} + \Gamma_n$. Then $H|_{T_n}$ is semiample & $(*) \Rightarrow H$ is nef & abundant $\therefore$ H is semiample by BPFT for dlt pairs. Thus $f_n : X_n \dashrightarrow Y_n$ is a good minimal model by 2.6.1.

__Lemma 6.2__ Suppose $\pi : (X, \Delta) \to U$ be a projective morphism to a smooth affine variety U, coeff $\Delta \in (0,1] \cap \mathbb{Q}$. Suppose (X, Δ) is log. smooth $/U$. If (X, Δ) has a good minimal model, then every fiber (X_u, Δ_u) does.

Pf: By previous trick, we reduce to the case $B_-(K_{X_u} + \Delta_u)$ contains no strata of Δ_u. Let A be ample $/U$. We apply [HX - 2.7].

$R(X/U, K_X + \Delta, K_X + \Delta + A) = \bigoplus_{\mathbb{N}^2} H^0(m_1(K_X + \Delta) + m_2(K_X + \Delta + A))$. (how?)

Since $K_X + \Delta$ & $K_X + \Delta + A$ both become semiample, after running an MMP for $K_X + \Delta + A$, this section ring then is f.g.

Thus the $(K_X + \Delta)$-MMP $\varphi : X \dashrightarrow Y$ with scaling of A terminates with a good MM for (X, Δ) over U. Since $B_-(K_{X_u} + \Delta_u)$ contains no strata of Δ_u, $3.1 \Rightarrow \pi_u : X_u \dashrightarrow Y_u$ is a semiample model of (X_u, Δ_u). $2.9.1 \Rightarrow (X_u, \Delta_u)$ has a good MM.

Thm 6.2: In the above situation, if one fiber (X_0, Δ_0) has a good MM, then (X, Δ) has a good MM $/U$ & every fiber has a good MM.

Pf: By 6.1, the generic fiber (X_η, Δ_η) has a good MM. So we may find a good MM of $\pi^{-1}(U_0)$ over an open subset $U_0 \subset U$. As (X, Δ) is log smooth $/U$, every stratum of $S = \lfloor \Delta \rfloor$ intersects $\pi^{-1}(U_0)$. [HX, 1.1]: (X, Δ) has a good MM over U. By the arguments of 6.2, every fiber has a good MM.

Lemma 6.3 Let $\pi : (X, \Delta) \to U$ as above, $\mathrm{coeff}(\Delta) \in (0, 1] \cap \mathbb{Q}$. If $\exists\, 0 \in U$ closed pt s.t (X_0, Δ_0) has a good MM, then the restriction $\pi_* \mathcal{O}_X(m(K_X + \Delta)) \longrightarrow H^0(\mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$ is surj $\forall\, m \in \mathbb{N}$ s.t $m\Delta$ is integral.

Rmk: Note that if $\lfloor \Delta \rfloor = 0$ (klt case), this is Theorem 4.2

Pf: WMA $m \geq 2$ (O.w., this is 2.3.4). Replacing U by a finite cover, WMA π is a contraction & strata of Δ have irred fibres over U. WMA U is affine by local nature of the problem. Suppose we know invariance of (log) plurigenera holds. Then $\pi_* O_X(m(K_X+\Delta))$ is a vector bundle on U which we may assume trivial by further shrinking U.
Also C-B-C: $\pi_* O_X(m(K_X+\Delta)) \otimes k(u) \xrightarrow{\sim} H^0(O_{X_u}(m(K_{X_u}+\Delta_u)))$ $\forall u \in U$. & $\pi_* O_X(---) \twoheadrightarrow \pi_*(---) \otimes k(u)$ (since it's globally gen'd) $\therefore \pi_* O_X(---) \twoheadrightarrow H^0(m(K_{X_u}+\Delta_u))$ $\forall u \in U$ & we're done. So ETS: Invariance of (log) plurigenera. For this we may assume U is a curve ($\because$ any two pts are joined by a curve).

Let $f_0 : Y_0 \to X_0$ be the birational morphism given by 2.8.3. It extends to a birational morphism $f: Y \to X$ which is a composition of smooth blowups of the strata of Δ.

Write $K_Y + \Gamma = f^*(K_X + \Delta) + E$ as usual. Then (Y, Γ) is log smooth & the fibres of cpts of Γ are irreducible, so Γ is irreducible & $H^0(\mathcal{O}_X(m(K_X+\Delta))) \simeq H^0(\mathcal{O}_Y(m(K_Y+\Gamma)))$ & $H^0(\mathcal{O}_{X_0}(m(K_{X_0}+\Delta_0))) \simeq H^0(\mathcal{O}_{Y_0}(m(K_{Y_0}+\Gamma_0)))$ (the latter is true b/c $E|_{Y_0}$ is f_0-exc $\because f$ extends f_0).

Replacing (X, Δ) by (Y, Γ), 2.8.3 $\Rightarrow$ if $\Theta_0 \doteq \Delta_0 - \Delta_0 \wedge N_\sigma(K_{X_0} + \Delta_0)$ then $\mathbf{B}_-(K_{X_0} + \Theta_0)$ contains no strata of Θ_0. $\exists!\ 0 \le \Theta \le \Delta$ s.t. $\Theta|_{X_0} = \Theta_0$. Now 1.2 $\Rightarrow K_X + \Delta$ is psef & so 2.3.3 $\Rightarrow$ $\Theta = \Delta - \Delta \wedge N_\sigma(K_X + \Delta)$. Now (X_0, Δ_0) has a good MM $\Rightarrow (X_0, \Theta_0)$ does $\overset{1.2}{\Rightarrow} (X, \Theta)$ has a good MM $/U \overset{HK\ 2.9}{\Longrightarrow}$ any run of the $(K_X + \Theta)$-MMP over U with ample scaling always terminates. 3.2 $\Rightarrow$ we may run the $(K_X+\Theta)$-MMP $f: X \dashrightarrow Y$ over U until we get a semiample model of the generic fiber. 3.1 $\Rightarrow f$ is an isom in a nbd of the generic pt of every nonklt center of $(X, X_0 + \Theta)$. [note: any $(K_X+\Theta)$-MMP$/U$ is also a $(K_X + \Theta + X_0)$-MMP over U]. Since any MMP over U terminates, we may continue this MMP until we get to a good MM/U without changing the fiber at 0.

Let $V \subset X \times Y$ be the graph. Then $V \longrightarrow X$ is an isom in a nbd of the generic pt of each nklt center of $(X, X_0 + \Theta)$.

We may find a log resoln $W \longrightarrow V$ of the strict transf of Θ & the exc p div of $V \longrightarrow Y$ which is an isom in a nbd of the generic pt of each nklt center of $(X, X_0 + \Theta)$.

$$\begin{tikzcd} & W \arrow[d] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "q"] & \\ & V \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

Write $K_W + \Phi + W_0 = p^*(K_X + X_0 + \Theta) + E$, where W_0 is the strict transf of X_0, Φ that of $\lfloor \Theta \rfloor$ & $\lceil E \rceil \geq 0$ as p is an isom in a nbd of the generic pt of each nklt centre of $(X, X_0 + \Theta)$.

We may also write $p^*((m-1)(K_X+\Theta)) = q^* f_*((m-1)(K_X+\Theta)) + F^{\geq 0}$

Possibly shrinking U, wma $X_0 \sim_{\mathbb{Q}} 0$ (b/c A_0 (affine curve) is torsion).

If we set $A := p^*(m(K_X+\Theta)) + E - F$, $L := \lceil A \rceil$, $C := \{-A\}$

then $L - W_0 = p^*(m(K_X+\Theta)) + E - F + C - W_0$

$= p^*(K_X+\Theta) + E + p^*((m-1)(K_X+\Theta)) - F + C - W_0$

$\sim_{\mathbb{Q}} K_W + \Phi + C + q^* f_*((m-1)(K_X+\Theta))$. (Easy to check)

$(W, \Phi + C)$ is lc since $(W, \Phi+C)$ is log smooth & $\Phi + C$ is a boundary. Since all rklt centers of $(W, \Phi+C)$ dominate U, Ambro injectivity theorem (see below) gives $H^1(\mathcal{O}_W(L-W_0)) \xhookrightarrow{\because W_0 \sim_{\mathbb{Q}} 0} H^1(\mathcal{O}_W(L))$

& so the restriction $H^0(\mathcal{O}_W(L)) \xrightarrow{(\dagger)} H^0(\mathcal{O}_{W_0}(L|_{W_0}))$ is surj.

Note that $\mathrm{Supp}(L - \lfloor q^* f_*(m(K_X+\Theta)) \rfloor) \not\supseteq W_0$ &

$L - \lfloor q^* f_*(m(K_X+\Theta)) \rfloor = \lceil A \rceil - \lfloor q^* f_*(m(K_X+\Theta)) \rfloor \geq \lceil A - q^* f_*(m(K_X+\Theta)) \rceil$
(easy to check)

$= \lceil E + \frac{1}{m-1} F \rceil \geq 0$

$\therefore |L| \supset |\lfloor q^* f_*(\cdots) \rfloor| \quad (*)$

We also have $|L| \subset |mp^*(K_X+\Delta) + \lceil E - F \rceil| \subset |mp^*(K_X+\Delta) + \lceil E \rceil|$

$= |m(K_X+\Delta)|$

$(\because \lceil E \rceil \geq 0$ is p-exc$)$

Let $q_0 : W_0 \to Y_0$ the restriction of q.

We have
$$
\begin{aligned}
|m(K_{X_0} + \Delta_0)| &= |m(K_{X_0} + \Theta_0)| \\
&= |m(K_{Y_0} + f_{0*}\Theta_0)| \\
&= |q_0^* m(K_{Y_0} + f_{0*}\Theta_0)| \\
&\underset{\substack{\text{by } (*) \\ \text{above}}}{\subset} |L|_{W_0}| \underset{\substack{\text{by } (\dagger) \\ \text{above}}}{=} |L|/_{W_0} \subset |m(K_X + \Delta)|_{X_0}
\end{aligned}
$$

$\therefore H^0(m(K_X + \Delta)) \twoheadrightarrow H^0(m(K_{X_0} + \Delta_0))$.

as desired. $\square$

<u>Ambro's injectivity theorem</u>: Let (X, B) be lc, $L \sim_{\mathbb{R}} K_X + B$, H a semiample Cartier divisor on X. Let $m_0 \geq 1$ & $s \in \Gamma(\mathcal{O}_X(m_0 H))$ s.t $s|_C \not\equiv 0$ $\forall$ lc center C of (X, B). Then the mult

$\otimes s : H^1(\mathcal{O}_X(L + mH)) \to H^1(\mathcal{O}_X(L + (m + m_0)H))$ is injective $\forall m \geq 1$.

— x —